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Minimum Fuel Spacecraft Reorientation

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Fuel optimal solutions for the reorientation of an inertially symmetric rigid spacecraft with independent three-axes controls are investigated. All possible optimal control strategies are identified. These include bang-bang solutions, finite-order singular arcs, and infinite-order singluar arcs. Higher order necessary conditions for optimality of finite-order singular arcs are presented. Numerical examples of fuel optimal solutions with fixed maneuver time are presented involving all of the theoretically possible control logics.

I. Introduction

OPTIMAL rigid body rotational maneuvers have been studied by many researchers in the context of spacecraft attitude control systems.[1-5] Minimum time problems with bounded control energy and/or minimum control effort problems subject to fixed maneuver time have been the main objectives in these studies.

In a recent study Bilimoria and Wie[1] were the first to successfully solve minimum time 180-deg rest-to-rest reorientation problems subject to bounded control torques through a rigorous optimal control approach. Later, Seywald and Kumar[6] gave a complete analysis of all possible control logics associated with the same dynamical system and performance index. Besides the nonsingular control solutions found in Ref. 1, finite- and infinite-order singular control strategies were derived and numerical examples involving these control logics were presented.

In the present paper, the authors investigate the problem of reorienting an inertially symmetric spacecraft with minimum fuel expenditure from given initial angular position and velocity to fully or partly prescribed final positions. Three bounded independent control torques are assumed with the control axes aligned along prescribed principal axes. The mass of fuel burned is ignored in that the moments of inertia are unchanged during the maneuver. Except for an additional mass differential equation, the dynamical system used here is the same as in Refs. 1 and 6.

The identification of all possible optimal control logics for the problem stated is emphasized. To avoid absolute values in the right-hand side of the fuel mass differential equation, each control torque is divided into its positive and its negative component, thus giving rise to six mathematical controls instead of the three physical controls. For each of the six mathematical controls, the possible optimal control logics are either bang-bang type, or of finite/infinite-order singular type.

Higher order necessary conditions for optimality of finite-order singular arcs are examined using Goh transformations of the associated accessory minimum problem. Numerical examples of optimal solutions with bang-bang structure as well as finite- and infinite-order singular arcs are presented.

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II. Dynamical System

Consider an inertially symmetric rigid spacecraft with the control axes aligned with prescribed principal axes. The Euler rotational equations of motion can easily be derived as

$$\frac{d\omega}{dt} = u \tag{1}$$

where

$$\omega^T = [\omega_1, \omega_2, \omega_3]$$

is the vector of angular velocities of the rigid body about the three body principal axes (shown in Fig. 1) represented in the body coordinate frame and t is the clock time. The three independent components of control vector

$$u^T = [u_1, u_2, u_3]$$

are confined to the fixed limits

$$-u_{i,\max} \le u_i \le +u_{i,\max} \tag{2}$$

In physical terms $u_i = T_i/I$, where T_i, $i = 1, 2, 3$, are the three independent control torques represented in the body coordinate frame and I denotes the moments of inertia of the spacecraft about the chosen principal axes. The kinematic equations of motion can be written as

$$\frac{dq}{dt} = \frac{1}{2}\Omega q \tag{3}$$

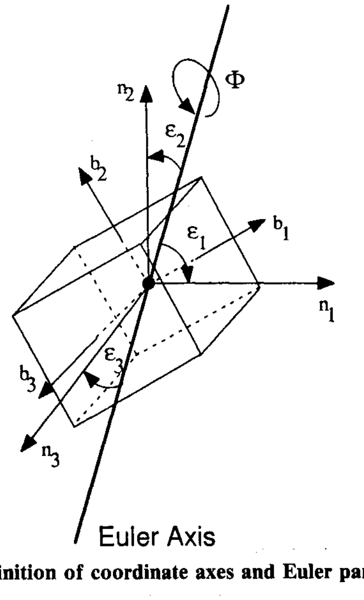

Fig. 1 Definition of coordinate axes and Euler parameters.

where

$$\Omega = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{4}$$

Here the Euler parameter vector

$$q^T = [q_0, q_1, q_2, q_3]$$

is used in favor over the Euler angles to prevent singularities and to simplify the kinematic differential equations. However, a drawback in the use of Euler parameter is the increase in the state vector dimension by 1.

The physical meaning of the quaternions is as follows:

$$q_0 = \cos(\Phi/2)$$

$$q_i = \cos \epsilon_i \sin(\phi/2); \qquad i = 1, 2, 3$$

where $(\epsilon_1, \epsilon_2, \epsilon_3)$ are the angles between the Euler axis and an inertially fixed axis orthogonal coordinate system (without loss of generality coincident with the body axis at initial time) as shown in Fig. 1. It may be noted that these are also the angles between the body axes and the Euler axis.

A principal axis rotation is defined as a rotation during which the Euler axis is always aligned with a prescribed body principal axis (also the same as a control axis).

In the course of this paper we will find it helpful to introduce the following matrices

$$D_1 := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{5}$$

$$D_2 := \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{6}$$

$$D_3 := \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

The benefit of these definitions lies in the fact that $\Omega = \omega_1 D_1 + \omega_2 D_2 + \omega_3 D_3$ along with the simple relations

$$\begin{aligned} D_i^T &= -D_i \\ D_i D_i &= -I \\ D_1 D_2 &= -D_3 \\ D_2 D_3 &= -D_1 \\ D_3 D_1 &= -D_2 \\ D_i D_j &= -D_j D_i, \quad i \neq j \end{aligned} \tag{8}$$

where $i, j \in \{1, 2, 3\}$.

III. Optimal Control Problem

For $c > 0$, let us consider the following optimal control problem [minimum-fuel spacecraft reorientation (MFSR)]:

$$\min_{(v,w) \in (PWC[t_0, t_f])^6} t_f + c \cdot m(t_f) \tag{9}$$

subject to the dynamical system

$$\dot{\omega} = v - w \tag{10}$$

$$\dot{q} = \tfrac{1}{2}\Omega q \tag{11}$$

$$\dot{m} = v_1 + v_2 + v_3 + w_1 + w_2 + w_3 \tag{12}$$

the initial conditions

$$\omega(0) = \omega^0, \qquad \omega^0 \in \boldsymbol{R}^3 \tag{13a}$$

given

$$q(0) = q^0, \qquad q^0 \in \boldsymbol{R}^4 \tag{13b}$$

given

$$m(0) = 0 \tag{13c}$$

the final conditions

$$\Psi\left[\omega(t_f), q(t_f)\right] = 0 \tag{14}$$

the control constraints

$$\begin{aligned} v_i &\in [0, u_{i,\max}], \qquad i = 1, 2, 3 \\ w_i &\in [0, u_{i,\max}], \qquad i = 1, 2, 3 \end{aligned} \tag{15}$$

and free final time t_f. Note that control vector u used in Eq. (1) is divided into its positive and negative components

$$u_i = v_i - w_i, \quad i = 1, 2, 3 \tag{16}$$

This is necessary to avoid nonsmooth appearance of control components in the right-hand side of Eq. (12). In Eq. (9) $PWC[t_0, t_f]$ denotes the set of all piecewise continuous functions on the interval $[t_0, t_f]$. The analysis in the remainder of this paper remains unchanged if we replace $PWC[t_0, t_f]$ by $L_\infty[t_0, t_f]$, the set of all essentially bounded measurable functions on the interval $[t_0, t_f]$ as the set of admissible control functions.

IV. Noncontrollability

From Eqs. (3) and (11) and the antisymmetry of Ω given in Eq. (4), it follows readily that $d/dt(q^T q) = 0$. Hence the mathematical model of the MFSR dynamics given by Eqs. (10–12) is not completely controllable. For this reason the function

$$\Psi : \begin{cases} \boldsymbol{R}^3 \times \boldsymbol{R}^4 & \rightarrow \boldsymbol{R}^k \\ \left[\omega(t_f), q(t_f)\right] & \mapsto \Psi\left[\omega(t_f), q(t_f)\right] \end{cases}$$

representing the final conditions (14) has to satisfy $k \leq 6$ and

$$\operatorname{rank}\left[\frac{\partial \Psi}{\partial q(t_f)}\right] \leq 3 \tag{17}$$

This represents a necessary condition for the existence of a nondegenerate solution of problem MFSR. It can be seen from the costate equations stated later in Eq. (20) that the lack of controllability in the q equations translates into the same lack of controllability in the λ_q equations. This is an important point to realize when it comes to setting up a consistent boundary value problem for generating numerical solutions.

V. Optimality Conditions

The Hamiltonian H is defined by

$$H = \lambda_\omega^T(v - w) + \tfrac{1}{2}\lambda_q^T \Omega q + \lambda_m \sum_{i=1}^{3} (v_i + w_i) \tag{18}$$

and the costate equations $\dot{\lambda}_x = -(\partial H/\partial x)^T, x \in \{\omega, q, m\}$ take the explicit form

$$\dot{\lambda}_\omega = -\frac{1}{2}\begin{bmatrix} q^T D_1^T \lambda_q \\ q^T D_2^T \lambda_q \\ q^T D_3^T \lambda_q \end{bmatrix} \tag{19}$$

$$\dot{\lambda}_q = -\tfrac{1}{2}\Omega^T \lambda_q \tag{20}$$

$$\dot{\lambda}_m = 0 \tag{21}$$

At each instant of time the Pontryagin minimum principle (see Refs. 7-10) requires the controls to be such that the Hamiltonian (18) is minimized. With the switching functions S_i, T_i defined by

$$S_i = \frac{\partial H}{\partial v_i} = \lambda_m + \lambda_{\omega_i}, \quad i = 1, 2, 3 \tag{22}$$

$$T_i = \frac{\partial H}{\partial w_i} = \lambda_m - \lambda_{\omega_i}, \quad i = 1, 2, 3 \tag{23}$$

we get the control logic

$$v_i = \begin{cases} 0 & \text{if } S_i > 0 \\ \text{singular} & \text{if } S_i \equiv 0 \\ u_{i,\max} & \text{if } S_i < 0 \end{cases} \tag{24}$$

$$w_i = \begin{cases} 0 & \text{if } T_i > 0 \\ \text{singular} & \text{if } T_i \equiv 0 \\ u_{i,\max} & \text{if } T_i < 0 \end{cases} \tag{25}$$

The singular cases $\lambda_{w_i} \equiv +\lambda_m$, $\lambda_{w_i} \equiv -\lambda_m$, are treated more explicitly in the sections hereafter.

At times t_s, where any of the control variables v_i, w_i switch between the possible control logics stated in Eqs. (24) and (25), a necessary condition for optimality is given by the continuity of the Hamiltonian

$$H|_{t_s^+} - H|_{t_s^-} = 0 \tag{26}$$

In practice this condition is used to determine the location of switching times. The transversality conditions associated with boundary conditions (14) and the cost function (9) are given by

$$\lambda_\omega(t_f)^T = \nu^T \frac{\partial \Psi}{\partial \omega(t_f)} \tag{27}$$

$$\lambda_q(t_f)^T = \nu^T \frac{\partial \Psi}{\partial q(t_f)} \tag{28}$$

$$\lambda_m(t_f) = c \tag{29}$$

where Ψ satisfies Eq. (17) and $\nu \in R^k$ is a constant multiplier vector. The optimal final time is determined by

$$H|_{t_f} = -1 \tag{30}$$

VI. Control Logic in Terms of u

For a physical understanding of the control logics given by Eqs. (22-25), it is helpful to restate these conditions in terms of control u defined by Eq. (16). We have for $i = 1, 2, 3$

$$u_i = \begin{cases} -u_{i,\max} & \text{if } \lambda_{\omega_i} > \lambda_m \\ \text{singular (neg)} & \text{if } \lambda_{\omega_i} \equiv \lambda_m \\ 0 & \text{if } \lambda_m > \lambda_{\omega_i} > -\lambda_m \\ \text{singular (pos)} & \text{if } \lambda_{\omega_i} \equiv -\lambda_m \\ +u_{i,\max} & \text{if } \lambda_{\omega_i} > -\lambda_m \end{cases}$$

This control logic is illustrated by Fig. 2.

VII. Transformation of the Costate Dynamics

For analyzing the singular control cases $S_i \equiv 0$ or $T_i \equiv 0$, and for a better understanding of the costate dynamics, we will find it helpful to apply the transformation of the costate dynamics introduced in Ref. 6. For $i = 1, 2, 3$ define

$$d_i := -\tfrac{1}{2} q^T D_i^T \lambda_q \tag{31}$$

Then, using Eqs. (11) and (20) along with the simple relations given in Eq. (8), it is easy to verify that $d^T := [d_1, d_2, d_3]$ satisfies the differential equation

$$\dot{d} = \hat{\Omega} d \tag{32}$$

with

$$\hat{\Omega} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{33}$$

This means the dynamics of d can be expressed completely in terms of ω and d itself. Rewriting Eq. (19) as

$$\dot{\lambda}_\omega = d \tag{34}$$

and noting that λ_q does not appear explicitly in switching functions S_i, T_i defined by Eqs. (22) and (23), it is clear that the state/costate system (10), (11), (12), (19), (20), and (21) can be substituted equivalently by the simpler system (10), (11), (12), (34), (32), and (21). In analyzing the singular control cases we will make extensive use of this transformation.

VIII. Singular Control

A control u appearing only linearly in the Hamiltonian is called singular on some nonzero time interval $[\tau_1, \tau_2]$ if the associated switching function $\Theta = \partial H/\partial u$ is identically zero on $[\tau_1, \tau_2]$. As Θ is independent of control u, successive differentiation of the identity $\Theta \equiv 0$ with respect to time is well defined and can be used to obtain additional conditions.

In the present optimal control problem we have six linearly appearing controls v_i, w_i, $i = 1, 2, 3$, and six associated switch-

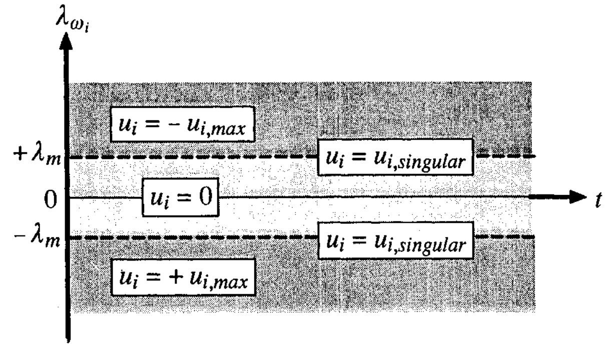

Fig. 2 Switching logic for control u.

ing functions defined by Eqs. (22) and (23), respectively. With $S := [S_1, S_2, S_3]^T$, $T := [T_1, T_2, T_3]^T$ we have

$$S_i = \lambda_m + \lambda_{\omega_i} \tag{35}$$

$$\dot{S} = d \tag{36}$$

$$\ddot{S} = \hat{\Omega} d \tag{37}$$

$$S^{(3)} = (\hat{\Omega}^2 + \dot{\hat{\Omega}}) d \tag{38}$$

and

$$T_i = \lambda_m - \lambda_{\omega_i} \tag{39}$$

$$\dot{T} = -d \tag{40}$$

$$\ddot{T} = -\hat{\Omega} d \tag{41}$$

$$T^{(3)} = -(\hat{\Omega}^2 + \dot{\hat{\Omega}}) d \tag{42}$$

where $\hat{\Omega}$ is given by Eq. (33). Conditions on singular optimal control for v_i or w_i are obtained from subsequently setting $S_i = 0$, $\dot{S}_i = 0$, etc., or $T_i = 0$, $\dot{T}_i = 0$, etc., respectively.

It is clear that all higher derivatives of S, T (if they exist) are linear in d. In particular, this implies that all derivatives of S, T are automatically zero only if $d = 0$.

Before we proceed with analyzing the singular control cases for problem MFSR, it should be noted that not all possible combinations of singular and nonsingular controls need to be considered. As $\lambda_m \equiv c > 0$ [see Eqs. (21) and (29)] it can be seen from Eqs. (24) and (25) that for every component $i \in \{1, 2, 3\}$ v_i and w_i cannot be singular at the same time. Consequently, only cases with one, two, or three mathematical controls being singular at the same time have to be considered.

Furthermore, it is clear that problem MFSR remains essentially unchanged if the coordinate axes are interchanged. This observation implies, for example, that in the case where only one control is singular, it can be assumed without loss of generality, that control v_1 or w_1 is singular, whereas v_2, w_2, v_3, w_3 are nonsingular.

In the following three sections a complete analysis of the singular control cases with one, two, and three controls being singular is presented. We consider two fundamentally different cases, namely $d_1^2 + d_2^2 + d_3^2 \neq 0$, which leads to finite-order singular control, and $d_1^2 + d_2^2 + d_3^2 = 0$, which leads to infinite-order singular control.

A. Finite-Order Singular Control

In this section we assume $d_1^2 + d_2^2 + d_3^2 \neq 0$. As we will see this assumption ensures that the singular control is always of finite order.

1. *One Control Singular*

Without loss of generality it can be assumed that controls v_2, w_2, v_3, w_3 are all nonsingular. Then we are left with the two possible cases that either v_1 is singular and w_1 is nonsingular, or, that w_1 is singular and v_1 is nonsingular. Explicitly, we get case 1 where v_1 is singular and w_1 is nonsingular:

$$\lambda_m + \lambda_{\omega_1} = 0 \tag{43a}$$

$$d_1 = 0 \tag{43b}$$

$$\omega_3 d_2 - \omega_2 d_3 = 0 \tag{43c}$$

$$\left[\omega_1\omega_2 + (v_3 - w_3)\right] d_2 + \left[\omega_1\omega_3 - (v_2 - w_2)\right] d_3 = 0 \tag{43d}$$

at the beginning of the singular arc, and

$$v_1 \equiv -2\omega_1 \frac{(v_2 - w_2)d_2 + (v_3 - w_3)d_3}{\omega_2 d_2 + \omega_3 d_3} \tag{44}$$

in the interior of the singular arc.

In case 2 w_1 is singular and v_1 is nonsingular:

$$\lambda_m - \lambda_{\omega_1} = 0 \tag{45a}$$

$$d_1 = 0 \tag{45b}$$

$$\omega_3 d_2 - \omega_2 d_3 = 0 \tag{45c}$$

$$\left[\omega_1\omega_2 + (v_3 - w_3)\right] d_2 + \left[\omega_1\omega_3 - (v_2 - w_2)\right] d_3 = 0 \tag{45d}$$

at the beginning of the singular arc, and

$$w_1 \equiv +2\omega_1 \frac{(v_2 - w_2)d_2 + (v_3 - w_3)d_3}{\omega_2 d_2 + \omega_3 d_3} \tag{46}$$

in the interior of the singular arc.

In case 1, Goh's necessary condition for singular control is satisfied if and only if $\omega_2 d_2 + \omega_3 d_3 > 0$. In case 2, the inequality sign is reversed and we get the condition $\omega_2 d_2 + \omega_3 d_3 < 0$.

In the remainder of this section we will give a short proof of the claims made for case 1. The claims made in case 2 follow analogously.

From Eqs. (35–37) we find that $S_1 = 0$, $\dot{S}_1 = 0$, and $\ddot{S}_1 = 0$, imply

$$\lambda_{\omega_1} + \lambda_m = 0 \tag{47}$$

$$d_1 = 0 \tag{48}$$

$$\omega_3 d_2 - \omega_2 d_3 = 0 \tag{49}$$

Using Eq. (48) in Eq. (38), $S_1^{(3)} = 0$ yields

$$\left[\omega_1\omega_2 + (v_3 - w_3)\right] d_2 + \left[\omega_1\omega_3 - (v_2 - w_2)\right] d_3 = 0 \tag{50}$$

Note that controls v_2, w_2, v_3, w_3 are assumed nonsingular and hence are identically constant, except possibly at a finite number of points. Further differentiation of Eq. (50) yields

$$v_1(\omega_2 d_2 + \omega_3 d_3) + 2\omega_1\left[(v_2 - w_2)d_2 + (v_3 - w_3)d_3\right] = 0 \tag{51}$$

With the assumption $d_1^2 + d_2^2 + d_3^2 \neq 0$, condition (48) ($d_1 = 0$) implies that Eq. (51) can be solved for control v_1 and we obtain the result stated in Eqs. (43) and (44). Kelley's necessary condition for optimality of singular control (see Refs. 11–14)

$$(-1)^q \frac{\partial}{\partial u_1}\left[\frac{\mathrm{d}^{2q}}{\mathrm{d}t^{2q}}\left(\frac{\partial G}{\partial u_1}\right)\right] \geq 0, \qquad q = 2$$

with $q \in N$ denoting the order of the singular control, yields $\omega_2 d_2 + \omega_3 d_3 \geq 0$. The same result can also be obtained from Goh's necessary condition (see Refs. 15 and 16).

The strengthened form of this inequality is immediately implied by assumption $d_1^2 + d_2^2 + d_3^2 \neq 0$ and Eq. (48) ($d_1 = 0$).

2. *Two Controls Singular*

This case can be excluded as long as $d_1^2 + d_2^2 + d_3^2 \neq 0$. Formally, conditions for a finite-order singular control can be obtained, but from Goh's necessary condition it follows that this control logic is always suboptimal. The analysis involving these nonoptimal singular control cases is very similar to the analysis presented in Ref. 6 for a minimum-time satellite reorientation problem. For further information the reader is referred to this paper.

3. *Three Controls Singular*

In this case, for each component $i = 1, 2, 3$, either v_i or w_i has to be singular. From Eqs. (36) and (40) it is clear that this always implies $d = 0$. Obviously this contradicts the assumption $d_1^2 + d_2^2 + d_3^2 \neq 0$. Hence, under the assumption $d_1^2 + d_2^2 + d_3^2 \neq 0$ the case of three controls being singular at the same time can be excluded.

B. Infinite-Order Singular Control

In this section we assume

$$d_1^2 + d_2^2 + d_3^2 = 0 \tag{52}$$

From Eqs. (36–38) and (40–42) it is immediately clear that, if assumption (52) is satisfied only at a single point, it is satisfied throughout the time interval. Furthermore, all derivatives of switching functions S_i, T_i, $i = 1, 2, 3$, remain constant throughout the time interval.

Specifically, in a singular control case, this implies that the switching function remains zero irrespective of the control. In fact, at each instant of time the numerical value of the singular control is arbitrary as long as control constraints (15) are satisfied and as long as the state vector is steered to the prescribed final position defined by Eq. (14).

1. One Control Singular

Without loss of generality it can be assumed that controls v_2, w_2, v_3, w_3 are all nonsingular. Then we are left with the two possible cases that either v_1 is singular and w_1 is nonsingular, or, that w_1 is singular and v_1 is nonsingular. Explicitly, we get case 1 where v_1 is singular and w_1 is nonsingular. Such control logic is characterized by the conditions

$$\begin{aligned} \lambda_{\omega_1} + \lambda_m &= 0 \\ d &= 0 \end{aligned} \tag{53}$$

and

$$\begin{aligned} \lambda_{\omega_2} &\neq \pm\lambda_m \\ \lambda_{\omega_3} &\neq \pm\lambda_m \end{aligned} \tag{54}$$

At each instant of time, the nonsingular controls v_2, w_2, v_3, w_3, and w_1, are determined from Eqs. (22–25). The singular control v_1 is arbitrary throughout the time interval as long as it satisfies control constraint (2) and as long as the state is steered to a final position that satisfies boundary condition (14).

In case 2 w_1 is singular and v_1 is nonsingular. This case is different from case 1 only in the sense that the roles of controls v_1 and w_1 are interchanged. Hence, the conditions characterizing the control logic are given by

$$\begin{aligned} \lambda_{\omega_1} - \lambda_m &= 0 \\ d &= 0 \end{aligned} \tag{55}$$

and

$$\begin{aligned} \lambda_{\omega_2} &\neq \pm\lambda_m \\ \lambda_{\omega_3} &\neq \pm\lambda_m \end{aligned} \tag{56}$$

2. Two Controls Singular

Without loss of generality it can be assumed that controls v_3, w_3 are nonsingular. Then the two singular controls are either v_1, v_2 or v_1, w_2 or w_1, v_2 or w_1, w_2. In the first case, i.e., with v_1, v_2 singular, we get explicitly

$$\lambda_{\omega_1} + \lambda_m = 0 \tag{57a}$$

$$\lambda_{\omega_2} + \lambda_m = 0 \tag{57b}$$

$$d = 0 \tag{57c}$$

The other cases work out analogously.

3. Three Controls Singular

Assuming that three controls are singular implies $H = 0$. Obviously, this is inconsistent with transversality condition (30). Hence this case can be excluded.

IX. Numerical Examples

The parameter $c > 0$ in Eq. (9) determines how much weight is put on minimizing the fuel consumption, possibly at the expense of a longer maneuver time t_f. For $c = 0$ we obtain the problem of minimum-time reorientation. This problem has first been successfully treated in Ref. 1, and a complete analysis of all possible control logics is given in Ref. 6.

For $c \to \infty$ we obtain pure fuel minimization with no penalty on the time that it takes to complete the maneuver. Typically, if the final time is left free, the maneuver time tends to infinity for such problems and a solution does not exist.

The problem of minimizing the fuel consumption for a prescribed reorientation within a fixed time (fixed-time problem) always has a solution. This can be verified easily by applying an existence theorem given by Lee and Markus.[9] In most cases such a problem formulation is equivalent to leaving the maneuver time free and specifying some positive penalty on the maneuver time and fuel consumption as done in the present paper (free-time problem). However, for the present problem there are cases where this equivalence, known as Mayer reciprocity, breaks down. To illustrate this, let us consider a 180-deg reorientation about the n_1 axis of a satellite spinning about the n_1 axis at initial time with a nonzero angular rate ω^0. Explicitly, let $t_0 = 0$ and consider the boundary conditions $\omega(t_0) = [10, 0, 0]$, $q(t_0) = [1, 0, 0, 0]$, $\omega(t_f) = [10, 0, 0]$, and $q(t_f) = [0, 1, 0, 0]$. Obviously, with no control action at all, the boundary conditions are reached at time $t_f = \pi/10$ and the fuel consumption assumes a global minimum. It is clear that some control actions may reduce the final time (e.g., first accelerate to $\omega_1 > 10$, then decelerate back to $\omega_1 = 10$) whereas other control actions may increase the final time (e.g., first decelerate to $\omega_1 < 10$, then accelerate back to $\omega_1 = 10$). In fact, for all $\epsilon > 0$ small enough, there are positive real numbers $\Delta t_1 > 0$, $\Delta t_2 > 0$, such that the final conditions $\omega(t_f) = [10, 0, 0]$, $q(t_f) = [0, 1, 0, 0]$, can be met for all prescribed final times $t_f \in [\pi/10 - \Delta t_1, \pi/10 + \Delta t_2]$ and fuel consumption (less or) equal to ϵ. But this implies that for small enough $\Delta t_2 > 0$ the optimal final time for the free time problem can never be $t_f = \pi/10 + \Delta t_2$. Clearly, there exists a solution with the same fuel consumption ϵ, and smaller final time $t_f = \pi/10 - \Delta t_2$. Hence, for this example, Mayer reciprocity breaks down. It is clear that for large enough $\Delta t_2 > 0$ new families of solutions can be found (e.g., reach the final conditions after two or more revolutions). Hence, this example also demonstrates that for fixed initial time and fixed initial states the set of reachable final states $K(t_f)$ may have a very complicated and nonintuitive structure.

To analyze the topology of the attainable sets $K(t_f)$ or to determine the structure of the optimal solutions for particular boundary conditions is beyond the scope of the present paper. Here it is only the aim to present the tools required to perform such work. That means it is intended to analyze the nature of the possible optimal subarcs from which the optimal control has to be synthesized. A complete theoretical analysis of the possible control logics is given in Secs. V–VIII. In the following subsections a numerical example is presented for each of these control logics. For all numerical calculations the control bounds $u_{i,\max}$, $i = 1, 2, 3$, used in Eq. (2) are set equal to 1. This choice is purely for convenience.

A. Nonsingular Control

The problem under consideration is that of a minimum-fuel rest-to-rest reorientation. Explicitly, this problem consists of Eqs. (9–12) and (15), subject to the initial conditions $\omega(0) = [0, 0, 0]$, $q(0) = [1, 0, 0, 0]$, the final condition $\omega(t_f) = [0, 0, 0]$, and a final condition associated with the prescribed final angular position. Even though the angular position is completely prescribed at final time not all Euler parameters can be prescribed at t_f. This is due to the lack of controllability in the dynamics of q discussed earlier in Sec. III. In the case of a 180-deg reorientation about the n_3 axis (see Fig. 3), the desired final value of state q is $q(t_f) = [0, 0, 0, 1]$. To obtain a consistent boundary value problem, only $q_0(t_f) = q_1(t_f) = q_2(t_f) = 0$ can be prescribed and $q_3(t_f)$ formally has to be considered free, which leads to the transversality condition $\lambda_{q_3}(t_f) = 0$.

If all four Euler parameters are prescribed explicitly at final time, then depending on the software used for solving boundary value problems, it may be impossible to obtain a solution, or at least the convergence behavior could be poor.

For the case of $c = 0.08$, Fig. 4 shows the time histories for controls, states, and costates. The optimal final time associated with this solution is $t_f = 3.405$ compared to $t_f = 3.243$ for the free final time problem ($c = 0$).

B. Finite-Order Singular Control

The only possible control logic involving finite-order singular control is given in Sec. VIII.A.1. Possibly after a permutation of the axis, it can be assumed without loss of generality that controls v_2, w_2, v_3, w_3 are all of bang-bang type. Then we are left with the two possible cases that either v_1 is singular and w_1 is nonsingular, or, that w_1 is singular and v_1 is nonsingular. With $c = 0.1$ in the cost criterion (9) a solution of the type of case 1 is obtained, i.e., v_1 singular, all other controls nonsingular, if we impose the following initial and final conditions:

$$\omega(0)^T = [1.158,\ 1.039,\ -0.498]$$

$$q(0)^T = [0.969,\ -0.185,\ -0.133,\ 0.098]$$

$$\omega(t_f)^T = [1.300,\ 0.739,\ -0.798]$$

$$q(t_f)^T = [1,\ 0,\ 0,\ 0]$$

The associated optimal controls as well as state and costate functions of time are given in Fig. 5.

C. Infinite-Order Singular Control

Consider problems (9–12) with initial conditions

$$\omega(0)^T = [0,\ 0,\ 0]$$

$$q(0)^T = [1,\ 0,\ 0,\ 0]$$

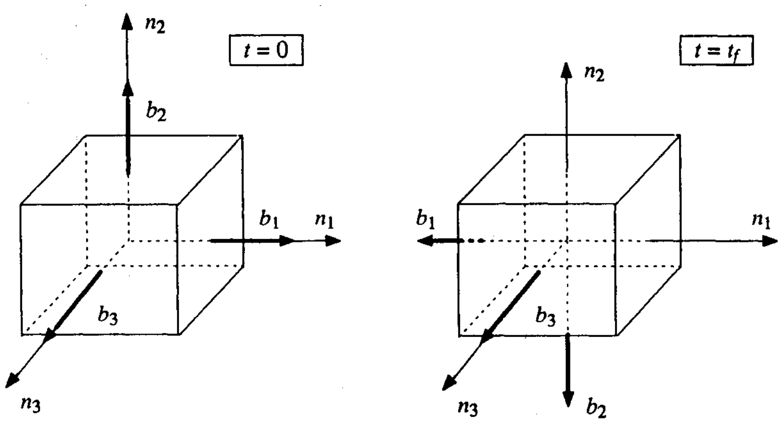

Fig. 3 180-deg reorientation about the body n_1 axis.

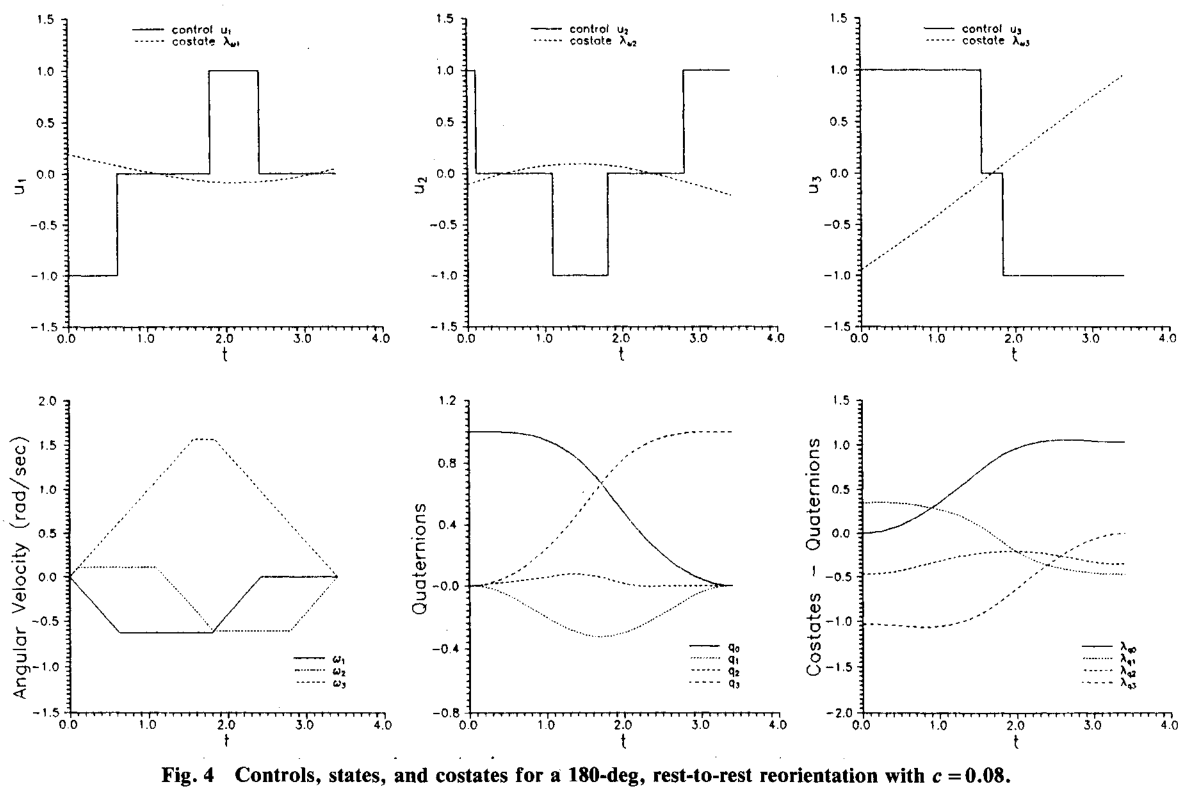

Fig. 4 Controls, states, and costates for a 180-deg, rest-to-rest reorientation with $c = 0.08$.

and let the final conditions on the angular velocity be given by

$$\omega(t_f) = [0.25,\ 0.25,\ 1]$$

Irrespective of the final conditions on the Euler parameters $q(t_f)$ it is then clear that a lower bound on the cost criterion (9) is given by

$$J[u] = t_f + c \cdot m(t_f) \geq 1 + 1.5 \cdot c$$

Clearly, a lower bound on the time that it takes to drive all angular velocities to their prescribed final values is

$$\max_{i=1,2,3} \{ |\omega_i(t_f) - \omega_i(0)| \} = 1$$

and a lower bound on the fuel that is required to achieve the prescribed changes in angular rates is

$$\sum_{i=1}^{3} |\omega_i(t_f) - \omega_i(0)| = 1.5$$

It can be verified easily that for some prescribed final angular positions there are whole families of solutions that satisfy the boundary conditions, all of them with the same cost equal to the lower bounds just given. One such case is obtained if we prescribe

$$q_0(t_f) = 0.96795$$

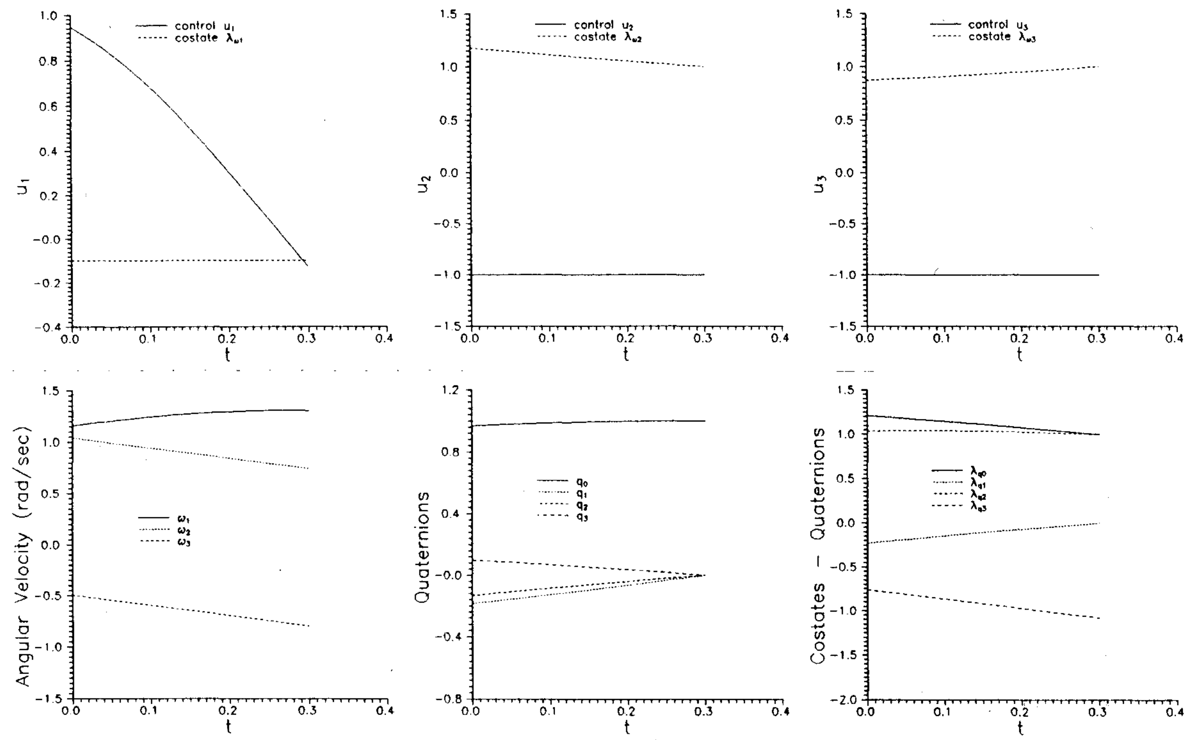

Fig. 5 Controls, states, and costates for a finite-order, singular control case with $c = 0.1$.

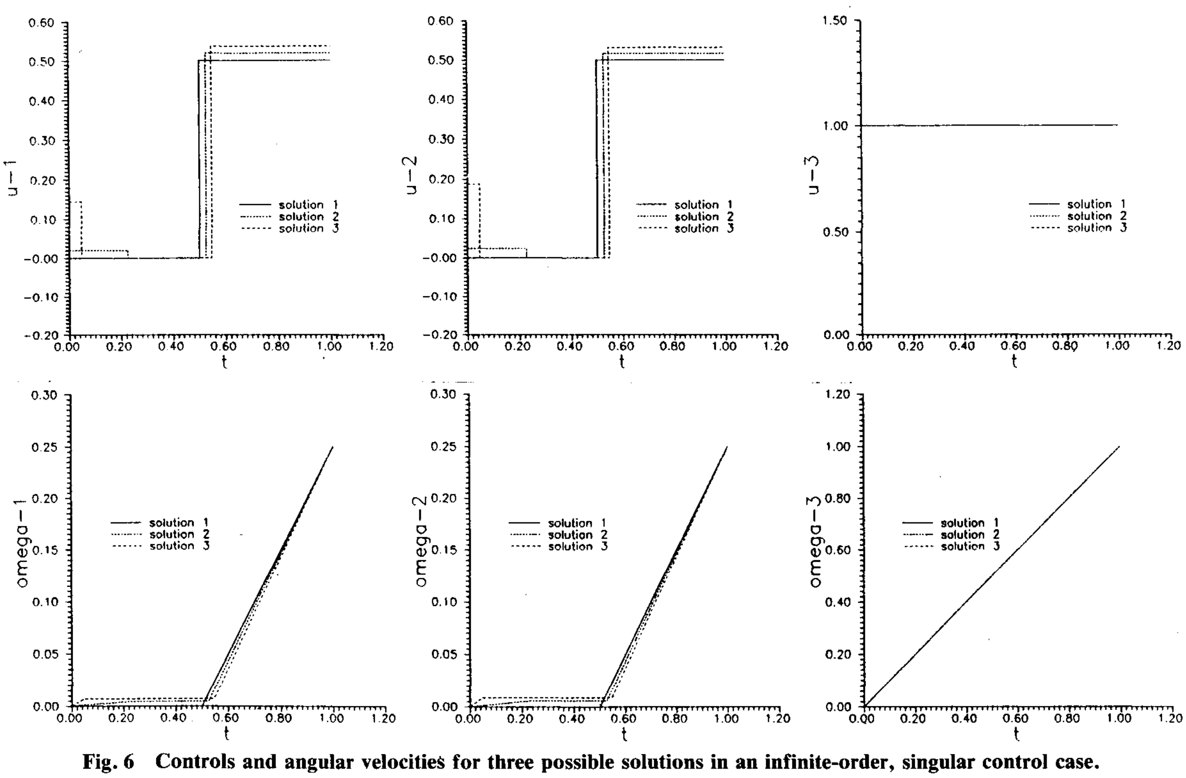

Fig. 6 Controls and angular velocities for three possible solutions in an infinite-order, singular control case.

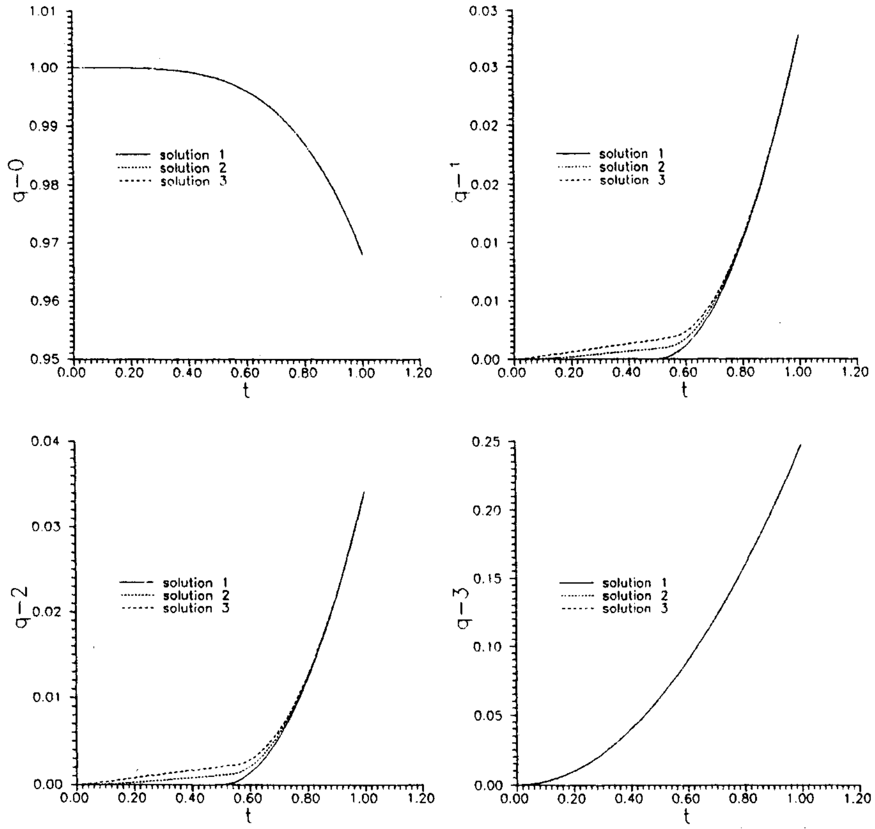

Fig. 7 Euler parameters for three possible solutions in an infinite-order, singular control case.

$$q_1(t_f) = 0.02770$$

$$q_2(t_f) = 0.03417$$

$$q_3(t_f) = 0.24727$$

The associated time histories of controls and states for three different solutions are given in Figs. 6 and 7.

Mathematically, the existence of families of solutions is reflected in infinite-order singular control. Along an optimal solution, the Hamiltonian is then independent of the control (singular control) and remains independent of the control no matter what control is chosen (infinite-order singular control).

For the preceding example, and for any $c > 0$, the costate functions of time are given as follows:

$$\lambda_{\omega_1} = -c \tag{58}$$

$$\lambda_{\omega_2} = -c \tag{59}$$

$$\lambda_{\omega_3} = -1 - c \tag{60}$$

$$\lambda_{q_0} = 0 \tag{61}$$

$$\lambda_{q_1} = 0 \tag{62}$$

$$\lambda_{q_2} = 0 \tag{63}$$

$$\lambda_{q_3} = 0 \tag{64}$$

$$\lambda_m = c \tag{65}$$

X. Conclusions

The minimum fuel reorientation problem for an inertially symmetric rigid spacecraft with three bounded independent control torques aligned with the principal axes is investigated. To avoid absolute values in the right-hand side of the fuel mass differential equation it is necessary to divide each control torque into its positive and its negative component, thus giving rise to six mathematical controls instead of the three physical controls. All possible control logics are clearly identified. These include bang-bang-type control and singular control of finite and infinite order. Numerical examples are provided for all cases.

The complex nature of minimum-time/minimum-fuel reorientation problems and the sometimes nonintuitive properties of the solutions is reflected in the following features:

1) Mayer reciprocity may not always hold between the free final time problems minimizing a linear combination of final time and consumed fuel (as discussed in the present paper) and fixed final time minimum-fuel problems.

2) For fixed-time, minimum-fuel problems the cost may not always decrease monotonically when the maneuver time is increased.

3) There are boundary conditions such that an infinite manifold of control functions of time leads to the same absolute minimum cost (infinite-order singular control). In this case the minimum principle does not provide conditions to determine the optimal control.

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